Discrete Mathematics 120 (1993) 59–73
North-Holland

Planar graphs with no 6-wheel minor

Bradley S. Gubser
Department of Mathematics, Hiram College, Hiram, OH 44234, USA

Received 25 March 1991
Revised 6 December 1991

Abstract

Gubser, B.S., Planar graphs with no 6-wheel minor, Discrete Mathematics 120 (1993) 59–73.

Tutte's wheels theorem states that the k-spoked wheel graphs, W_k, are the basic building blocks for the collection of simple, 3-connected graphs. Therefore it is of interest to examine the structure of the graphs that do not have a minor isomorphic to W_k for small values of k. Dirac determined that the graphs having no W_3-minor are the series-parallel networks. An easy consequence of Tutte's wheels theorem is that W_3 is the only simple, 3-connected graph that has a W_3-minor and no W_4-minor. Oxley characterized the graphs that have a W_4-minor and no W_5-minor. This paper characterizes the planar graphs that have a W_5-minor and no W_6-minor. A best-possible upper bound on the number of edges of such a graph is also determined.

1. Introduction

This paper states and proves a result on the structure of the simple, 3-connected, planar graphs that have no minor isomorphic to W_6, the 6-spoked wheel. The motivation for the problem was provided by other existing results of this type. For example, Dirac [2] determined that the graphs that have no minor isomorphic to W_3 are the series-parallel networks. This paper also describes the structure of all planar graphs that have no minor isomorphic to W_6. A bound on the number of edges of a simple planar graph having no minor isomorphic to W_6 is also determined.

Our notation will follow Bondy and Murty [1]. We allow a graph to have loops and 2-cycles; a *simple graph* has no loops or 2-cycles. We let $V(G)$ and $E(G)$ denote the sets of vertices and edges, respectively, of G. Let X be a subset of $E(G)$. We denote the deletion of X from G and the contraction of X from G by $G\backslash X$ and G/X, respectively. The graph G is said to be an *extension* of $G\backslash X$ and a *coextension* of G/X. The terms

Correspondence to: Bradley S. Gubser, Department of Mathematics, Hiram College, Hiram, OH 44234, USA.

single-element extension and *single-element coextension* will denote an extension and coextension, respectively, for which $|X|=1$. If X and Y are disjoint subsets of $E(G)$, then $G\backslash X/Y$ is called a *minor* of G. The graph G has an *H-minor* if G has a minor isomorphic to graph H.

The k-spoked wheel, denoted by W_k, is a simple graph consisting of a k-cycle, called the *rim*, and a vertex, called the *hub*, adjacent to each vertex of the k-cycle. We call the edges that join the rim to the hub the *spokes* of the wheel.

Let G_1 and G_2 be graphs having disjoint vertex sets. The *disjoint union* of G_1 and G_2 is the graph with vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)$. For $i=1$ and 2, let u_i be a vertex of G_i. The graph obtained by identifying u_1 and u_2 as a new vertex u is called the 1-*sum* of G_1 and G_2 with respect to vertices u_1 and u_2. For $i=1$ and 2, let u_iv_i be an edge of G_i. The graph $P((G_1, u_1v_1), (G_2, u_2v_2))$, called a *parallel connection* of G_1 and G_2 with respect to basepoints u_1v_1 and u_2v_2, is formed as follows: For $i=1$ and 2, delete u_iv_i from G_i. Identify the vertices u_1 and u_2 as a new vertex u and identify the vertices v_1 and v_2 as a new vertex v. Finally, join u and v by a new edge p. If, for all i in $\{1,2\}$, G_i has three or more edges and u_iv_i is neither a loop nor an isthmus, then $P((G_1, u_1v_1), (G_2, u_2v_2))\backslash p$ is called a 2-*sum* of G_1 and G_2. The following are two basic properties of 2-sum.

Proposition 1.1. *A graph G is simple, 2-connected, but not 3-connected if and only if G is a 2-sum of simple, 2-connected graphs G_1 and G_2 each having three or more vertices. Also, the 2-sum of G_1 and G_2 has a minor isomorphic to each of G_1 and G_2.*

Proposition 1.2. *If G is a 2-sum of graphs G_1 and G_2 and G has a minor isomorphic to a simple, 3-connected graph H, then G_1 or G_2 has a minor isomorphic to H.*

Suppose G has a W_k-minor for some $k\geqslant 3$. Then there is a sequence of graphs $H_1, H_2, \ldots, H_n$ such that $W_k\cong H_1$, $G\cong H_n$, and, for all i in $\{2, 3, \ldots, n\}$, there is an edge e_i of H_i such that either $H_i\backslash e_i$ or H_i/e_i is isomorphic to H_{i-1}. It would be beneficial to know when the intermediate graphs, $H_1, H_2, \ldots, H_n$ all have a certain property. In particular, 'the splitter theorem for graphs', a result of Seymour [6] and, independently, Negami [3], gives conditions for when the intermediate graphs may be chosen to be simple and 3-connected.

Theorem 1.3. *Let G and H be simple and 3-connected, each having at least four edges. Suppose G has an H-minor and, for all k, G is not isomorphic to W_k. Then there is an edge e of G such that one of $G\backslash e$ and G/e is simple, 3-connected, and has an H-minor.*

The following corollary is easily derived from the splitter theorem.

Corollary 1.4. *Let G be simple and 3-connected. Let k be the largest integer such that G has a W_k-minor. Then, for some $n\geqslant 1$, there is a sequence of simple, 3-connected graphs*

$H_1, H_2, \ldots, H_n$ such that $W_k \cong H_1$, $G \cong H_n$, and, for all i in $\{2, 3, \ldots, n\}$, H_i is a single-element extension or a single-element coextension of H_{i-1}.

To apply this corollary, we must be able to determine the single-element extensions and single-element coextensions of a given graph. Furthermore, these single-element extensions and single-element coextensions must be simple and 3-connected graphs. Such extensions and coextensions are called *non-trivial* single-element extensions and coextensions. Suppose H is both simple and 3-connected. A non-trivial single-element extension of H is found by joining two distinct non-adjacent vertices of H by a new edge. If G is a non-trivial single-element coextension of H, then G is obtained by *splitting* a vertex v of H, of degree four or more, into two adjacent vertices v' and v'' such that $G/v'v'' \cong H$ and both v' and v'' have degree at least three in G.

Suppose $\mathscr{C}$ is a collection of graphs that is closed with respect to taking minors. A graph H is a *splitter* for $\mathscr{C}$ if whenever G is in $\mathscr{C}$ and G has an H-minor, then $G \cong H$ or G is not both 3-connected and simple. The characterization of all simple, 3-connected, planar graphs having no W_6-minor consists of identifying the splitters for this class of graphs. Each of the splitters will also be simple, 3-connected, and planar.

2. The main theorem

As stated earlier, Dirac [2] showed that the graphs that have no W_3-minor are the series-parallel networks. From Tutte's wheels theorem [7] it is elementary to show that the only simple, 3-connected graph that has no W_4-minor is W_3. The following theorem of Oxley [5] characterizes the simple, 3-connected graphs that do not have a W_5-minor. The graphs H_6, Q_3, $K_{2,2,2}$, H_7, $K_{3,k}$, $K'_{3,k}$, $K''_{3,k}$ and $K'''_{3,k}$, which are referred to in the theorem, are illustrated in Fig. 1.

Theorem 2.1. *Let G be a graph. Then G is simple, 3-connected, and has no W_5-minor if and only if:*

(i) *G is isomorphic to a simple, 3-connected minor of H_6, Q_3, $K_{2,2,2}$, or H_7; or*

(ii) *for some $k \geqslant 3$, G is isomorphic to one of $K_{3,k}$, $K'_{3,k}$, $K''_{3,k}$, $K'''_{3,k}$.*

The main result of this paper characterizes the simple, 3-connected graphs that have no W_6-minor and are planar. Originally an attempt was made to classify all simple, 3-connected graphs having no W_6-minor; however, the large number of cases involved made this too difficult. Oporowski, Oxley, and Thomas [4] have shown that, for $k \geqslant 3$, there are only finitely many simple, 3-connected, planar graphs having no W_k-minor. This provided the necessary motivation to seek a characterization for the simple, 3-connected, planar graphs having no W_6-minor. Moreover, since every simple, 3-connected, planar graph with no W_6-minor is a minor of a maximal such graph, determining the splitters will characterize the class of simple, 3-connected, planar

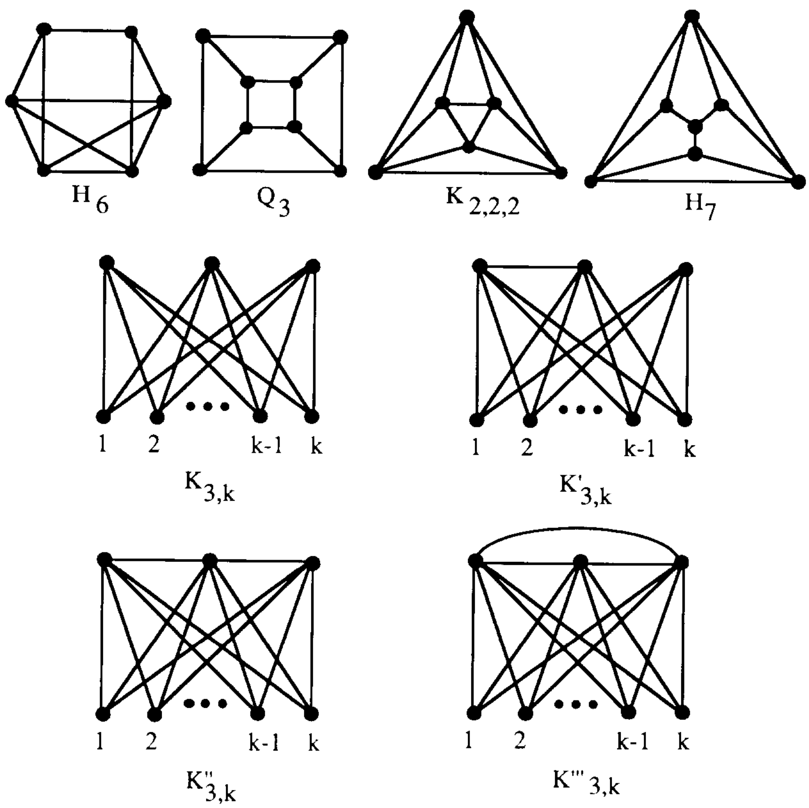

Fig. 1. Graphs H_6, Q_3, $K_{2,2,2}$, H_7, $K_{3,k}$, $K_{3,k}$, $K'_{3,k}$, $K''_{3,k}$, and $K'''_{3,k}$.

graphs having no W_6-minor. We now state the main theorem of this paper. The proof appears in Section 3.

Theorem 2.2. *Let G be a simple, 3-connected, planar graph that does not have a W_6-minor. Then G is isomorphic to a simple, 3-connected, planar minor of one of the thirty-eight graphs listed in Fig. 2.*

3. Proof of the Theorem 2.2

We must determine the splitters for the class of simple, 3-connected, planar graphs that have no W_6-minor. Let G be a splitter. We first show that G has a W_5-minor.

Suppose G does not have a W_5-minor. Then, by Theorem 2.1 and the fact that G is planar, G is isomorphic to a minor of H_6, Q_3, $K_{2,2,2}$, or H_7. It is routine to check that all planar minors of H_6 are isomorphic to a minor of Q_3 or $K_{2,2,2}$. Also, it is not difficult to show that if H is in $\{Q_3, K_{2,2,2}, H_7\}$, then there is a simple, 3-connected, planar graph H' such that (i) H' has a W_5-minor; (ii) H' does not have a W_6-minor; and (iii) for some e in $E(H')$, either $H'\backslash e = H$ or $H'/e = H$. Therefore H, and thus G, is not a splitter, a contradiction. We conclude that G has a W_5-minor.

Since G and W_5 are simple and 3-connected, we may apply Corollary 1.4. Not only must the graphs H_i specified in Corollary 1.4 be simple and 3-connected, but each must also be planar.

In the remainder of this section, the adjective *good* will describe a graph that is both simple and 3-connected.

The unique planar graph that has six vertices, ten edges, and a W_5-minor is W_5 (see Fig. 3). Assign the number $G(6, 10, 1)$ to this graph. In general, we shall use the following convention for keeping track of the good planar graphs that have no W_6-minor: $G(h, j, k)$ will denote the kth such graph having h vertices and j edges. We now determine the good planar graphs that have 11 edges and a W_5-minor but do not have a W_6-minor. Such a graph is either a good extension or a good coextension of $G(6, 10, 1)$. Note that any good extension of $G(6, 10, 1)$ is isomorphic to the graph $G(6, 11, 1)$, obtained by adding the edge v_1v_4 to $G(6, 10, 1)$. A good coextension of $G(6, 10, 1)$ is isomorphic to the graph $G(7, 11, 1)$, also depicted in Fig. 3. For clarity, the vertex v_i will be labelled i in all the figures.

Note that the graph $G(7, 11, 1)$ is isomorphic to the planar dual of $G(6, 11, 1)$. Whitney [9] showed that a simple, 3-connected, planar graph has a unique embedding in the plane. Thus the planar dual of $G(6, 11, 1)$ is uniquely determined. Suppose h, j, and k are given. Then it is not difficult to show that $G(h, j, k)^*$, the planar dual of $G(h, j, k)$, has a W_5-minor and no W_6-minor. Since $G(h, j, k)^*$ has $j - h + 2$ vertices and j edges, we will embed the concept of planar duality into our graph numbering system

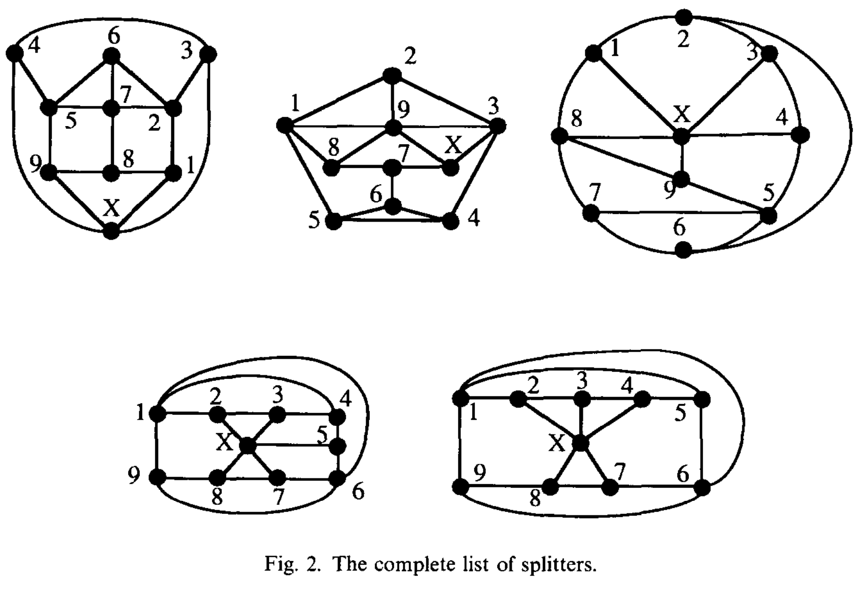

Fig. 2. The complete list of splitters.

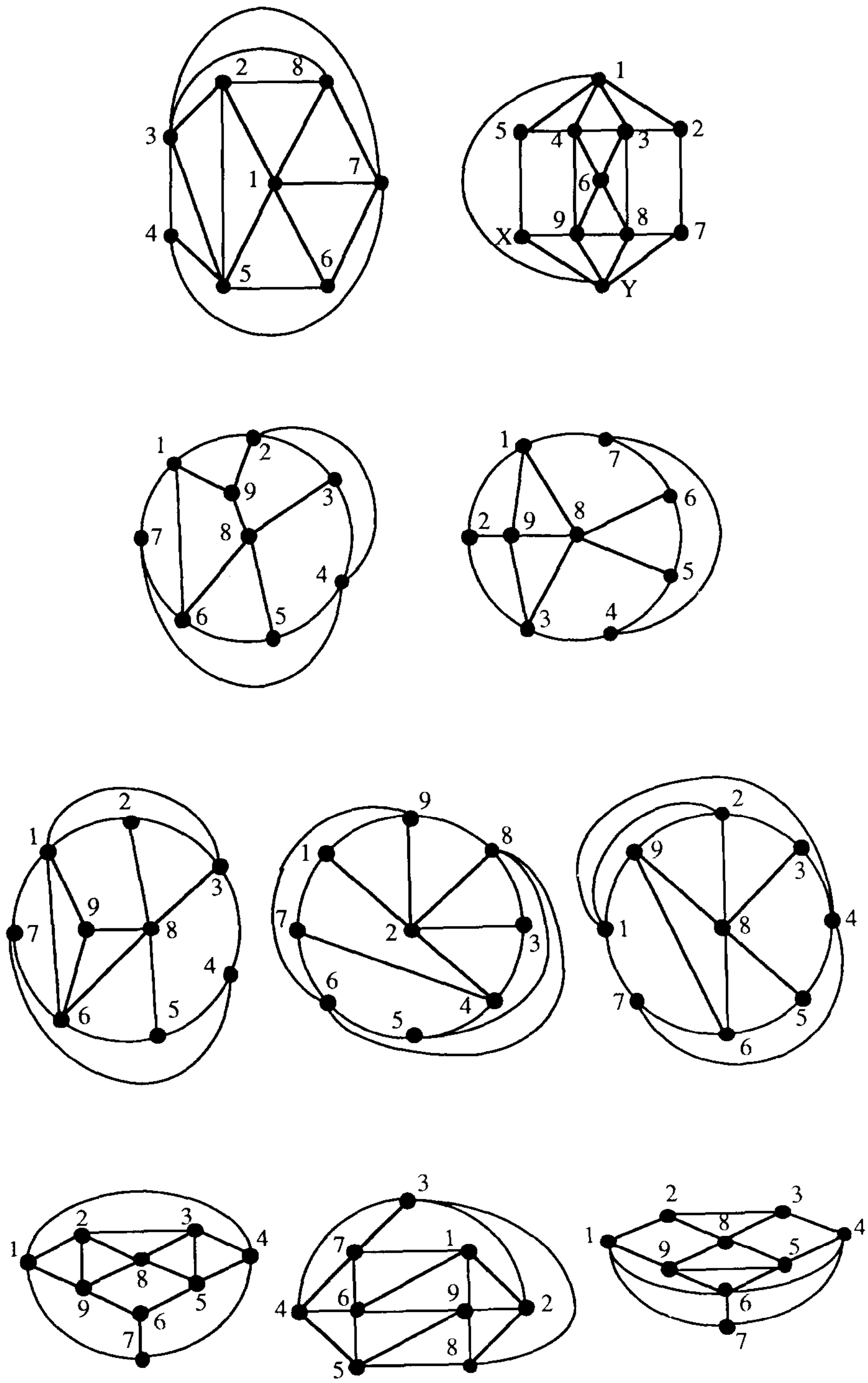

Fig. 2. Continued.

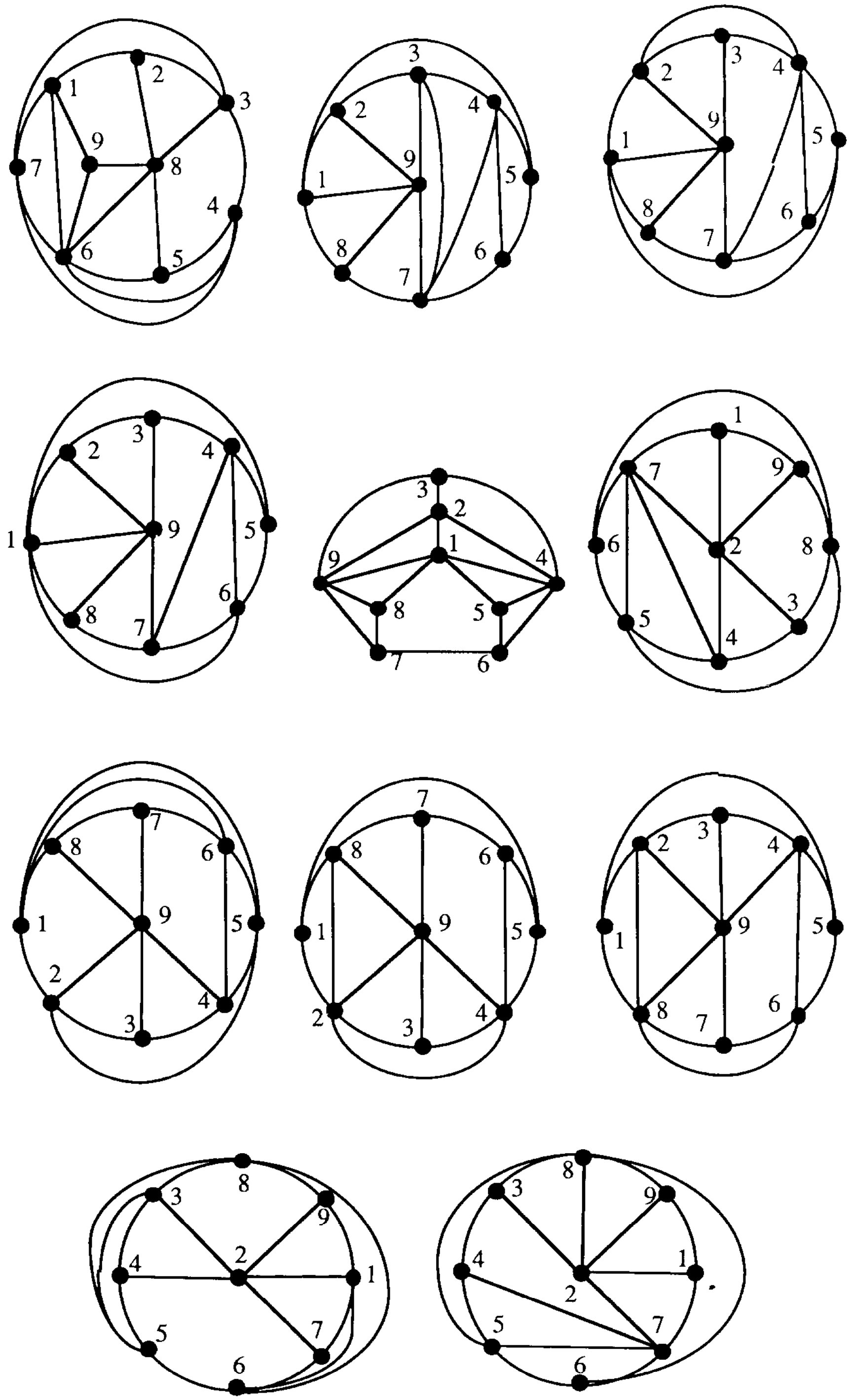

Fig. 2. Continued.

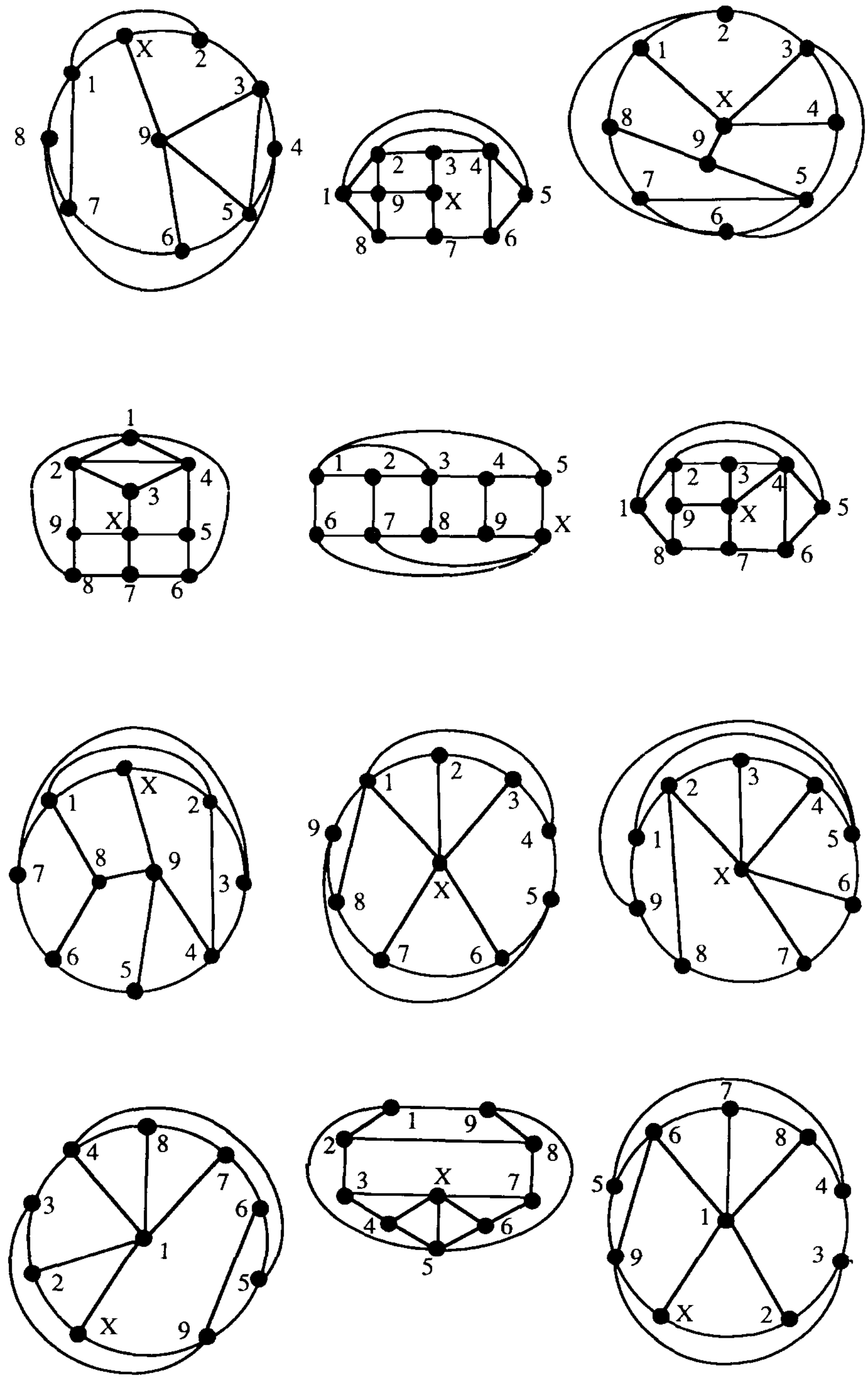

Fig. 2. Continued.

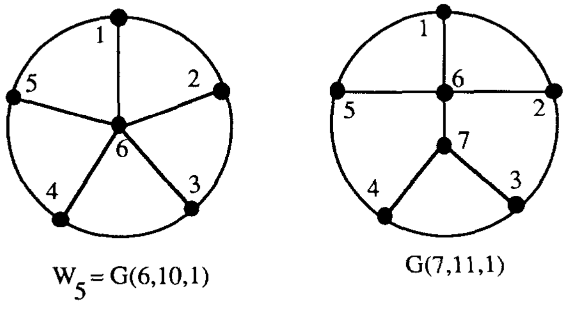

Fig. 3. Graphs $G(6,10,1)$ and $G(7,11,1)$.

by labelling the graph $G(h,j,k)^*$ as $G(j-h+2,j,k)$ whenever $h \neq (j+2)/2$. When h equals $(j+2)/2$, both $G(h,j,k)$ and $G(h,j,k)^*$ have the same number of vertices. In that case, the numbering system will not identify planar dual graphs.

Let H be a member of $\mathscr{G}(h,j)$, the collection of graphs $G(h,j,k)$. Suppose H is not isomorphic to W_5. Then H is a good extension of a member of $\mathscr{G}(h,j-1)$ or H is a good coextension of a member of $\mathscr{G}(h-1,j-1)$. Suppose H is a good coextension of H', a member of $\mathscr{G}(h-1,j-1)$. There is another way to view the relationship between H and H'. The graph $(H')^*$ is a member of $\mathscr{G}(j-h+2,j-1)$ while H^* is a member of $\mathscr{G}(j-h+2,j)$. It is elementary to show (see Welsh [8]) that H^* is an extension of $(H')^*$. Therefore H is the planar dual of some good extension of $(H')^*$. Thus $\mathscr{G}(h,j)$ is the set of graphs $G(h,j,k)$ such that $G(h,j,k)$ is a good extension of some member of $\mathscr{G}(h,j-1)$ together with the set of graphs $G(h,j,k)$ such that $G(h,j,k)^*$ is a good extension of some member of $\mathscr{G}(j-h+2,j-1)$. This is the method we will use to determine the graphs of $\mathscr{G}(h,j)$. For brevity, we will use the term *dual* to refer to the planar dual.

Since a planar graph has a unique embedding in the plane, we only consider adding an edge to $G(h,j,k)$ that would join two vertices that lie on a common face. For example, we are able to conclude that $G(6,11,1)+v_2v_5$ is non-planar without finding a K_5- or $K_{3,3}$-minor of $G(6,11,1)+v_2v_5$.

Let $G(6,12,1)$ denote the graph obtained by adding the edge v_1v_3 to $G(6,11,1)$. The only other edge which can be added to $G(6,11,1)$ is v_2v_4. This results in a graph which is isomorphic to $G(6,12,1)$. Hence, up to isomorphism, there is only one good planar graph that has exactly six vertices, 12 edges, and does not have a W_6-minor. Therefore there is only one good planar graph that has exactly eight vertices, 12 edges, and no W_6-minor. This graph, $G(8,12,1)$, is the dual of $G(6,12,1)$. It is depicted in Fig. 4.

We now determine the good planar graphs that have exactly seven vertices, 12 edges, and no W_6-minor. We begin by finding all good planar extensions of $G(7,11,1)$. There are nine edges that can be added to $G(7,11,1)$; however, up to isomorphism, this yields only five different graphs. For k in $\{1,2,3,4,5\}$, the graph $G(7,12,k)$ is found by adding the edge v_1v_4, v_3v_5, v_2v_5, v_4v_6, or v_5v_7, respectively, to $G(7,11,1)$. If e is v_1v_3, v_2v_4, v_3v_6, or v_2v_7, then $G(7,11,1)+e$ is isomorphic to $G(7,12,1)$, $G(7,12,2)$, $G(7,12,4)$,

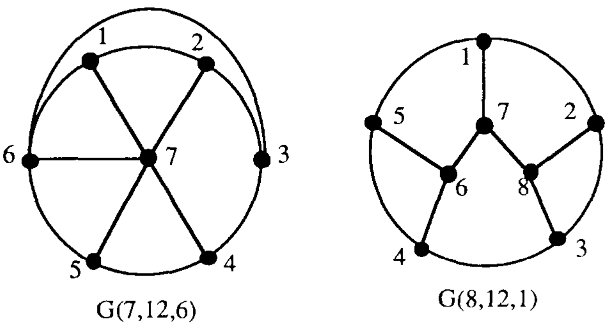

Fig. 4. Graphs $G(7,12,6)$ and $G(8,12,1)$.

or $G(7,12,5)$. For each isomorphism, the mapping of the vertices of $G(7,11,1)+e$ to the vertices of $G(7,12,k)$ is obvious. The notation used to number the graphs of $\mathscr{G}(7,12)$ does not include the concept of duality. This is because, for all G in $\mathscr{G}(7,12)$, both G and G^* have seven vertices. Therefore $\mathscr{G}(7,12)$ is equal to $\bigcup_{k=1}^{5}\{G(7,12,k), G(7,12,k)^*\}$. For $k=1,2,3$, or 4, the graph $G(7,12,k)$ is isomorphic to its dual. The dual of the graph $G(7,12,5)$ is labeled $G(7,12,6)$. The graph $G(7,12,6)$ is also illustrated in Fig. 4. Thus there are exactly six non-isomorphic good planar graphs that have exactly seven vertices, 12 edges and no W_6-minor. Hence, altogether, there are eight non-isomorphic good planar graphs that have 12 edges and no W_6-minor.

To determine the good planar graphs that have exactly 13 edges and no W_6-minor, we note that every face of $G(6,12,1)$ is a 3-cycle, therefore there is no planar extension of $G(6,12,1)$. Table 1 specifies extensions of each member of $\mathscr{G}(7,12)$. We interpret the entries of this table in the following manner. The first row states that $G(7,12,1)+v_1v_3=G(7,13,1)$. The first row having a $G(7,12,3)$ in column one states that the graph $G(7,12,3)+v_4v_6$ is isomorphic to $G(7,13,7)$ and the permutation that maps the indices of the vertices of $G(7,12,3)+v_4v_6$ to the indices of the vertices of $G(7,13,7)$ is (14657) (23). The last row of Table 1 states that $G(7,12,6)+v_3v_7$ has a W_6-minor. Such a minor is found by taking the cycle $v_1v_2, \ldots, v_6$ as the rim of W_6, the vertex v_7 as the hub, and the edges v_iv_7 for i in $\{1,2,\ldots,6\}$ as the spokes.

To find all good planar graphs that have exactly 13 edges and no W_6-minor, it remains to determine all such good planar graphs that have exactly eight vertices. A graph of this type is either a good extension of the graph $G(8,12,1)$ or the dual of $G(7,13,k)$ for some k. There are fourteen edges that may be added to $G(8,12,1)$. These edges yield only four non-isomorphic graphs. To illustrate this, observe that if both e and f lie in $\{v_2v_5, v_6v_8\}$, $\{v_2v_4, v_4v_8, v_3v_5, v_3v_6\}$, $\{v_5v_7, v_1v_8, v_1v_6, v_2v_7\}$, or $\{v_1v_4, v_4v_7, v_3v_7, v_1v_3\}$, then $G(8,12,1)+e\cong G(8,12,1)+f$. We shall use the first element in each set to define the graphs $G(8,13,1)$, $G(8,13,3)$, $G(8,13,4)$, and $G(8,13,7)$, respectively. The numbering scheme appears strange; however, these identification numbers reflect the fact that the graph $G(8,13,k)$ is isomorphic to the dual of

Table 1
Extensions of the graphs in $\mathscr{G}(7, 12)$

Graph H	Edge e	Graph $H+e$	Verification
$G(7, 12, 1)$	v_1v_3	**G(7, 13, 1)**	
	v_5v_7	**G(7, 13, 2)**	
	v_2v_4	**G(7, 13, 3)**	
	v_4v_6	**G(7, 13, 4)**	
	v_3v_6	**G(7, 13, 5)**	
	v_2v_7	**G(7, 13, 6)**	
$G(7, 12, 2)$	v_5v_7	**G(7, 13, 7)**	
	v_3v_6	**G(7, 13, 8)**	
	v_4v_6	$G(7, 13, 3)$	(1364) (725)
	v_1v_3	$G(7, 13, 3)$	(25) (34)
	v_2v_5	$G(7, 13, 7)$	(14) (27) (36)
	v_2v_7	$G(7, 13, 6)$	(13) (45) (67)
$G(7, 12, 3)$	v_4v_6	$G(7, 13, 7)$	(14657) (23)
	v_3v_6	$G(7, 13, 7)$	(1427) (365)
	v_5v_7	$G(7, 13, 7)$	(13) (45) (67)
	v_2v_7	$G(7, 13, 7)$	(13524) (67)
	v_3v_5	$G(7, 13, 7)$	(14) (27) (36)
$G(7, 12, 3)$	v_2v_4	$G(7, 13, 7)$	(25) (34)
$G(7, 12, 4)$	v_2v_7	**G(7, 13, 9)**	
	v_3v_5	$G(7, 13, 3)$	(1364) (725)
	v_1v_4	$G(7, 13, 4)$	identity
	v_1v_3	$G(7, 13, 5)$	(25) (34)
	v_2v_5	$G(7, 13, 7)$	(14657) (23)
	v_2v_4	$G(7, 13, 8)$	(25) (34)
	v_3v_6	W_6	H: 6; C: 123745
$G(7, 12, 5)$	v_1v_4	$G(7, 13, 2)$	identity
	v_3v_5	$G(7, 13, 7)$	identity
	v_2v_5	$G(7, 13, 7)$	identity
	v_1v_3	$G(7, 13, 6)$	(25) (34)
	v_2v_4	$G(7, 13, 6)$	(13524) (67)
	v_3v_6	$G(7, 13, 9)$	(25) (34)
	v_2v_7	$G(7, 13, 9)$	(13) (45) (67)
$G(7, 12, 6)$	v_2v_4	$G(7, 13, 1)$	(1237)
	v_1v_3	$G(7, 13, 3)$	(2536) (47)
	v_3v_5	$G(7, 13, 3)$	(1362745)
	v_2v_6	$G(7, 13, 4)$	(157432)
	v_4v_6	$G(7, 13, 4)$	(174) (23)
	v_3v_7	W_6	H: 7; C: 123456

$G(7,13,k)$. For $k=2,5,6,8$, and 9, the graph $G(8,13,k)$ is illustrated in Fig. 5. Thus there are a total of 18 non-isomorphic good planar graphs that have exactly 13 edges and no W_6-minor.

We now use Corollary 1.4 to determine the good planar graphs that have exactly 14 edges and no W_6-minor. Then we apply Corollary 1.4 again to determine the good planar graphs that have exactly 15 edges and no W_6-minor, and so on. From the

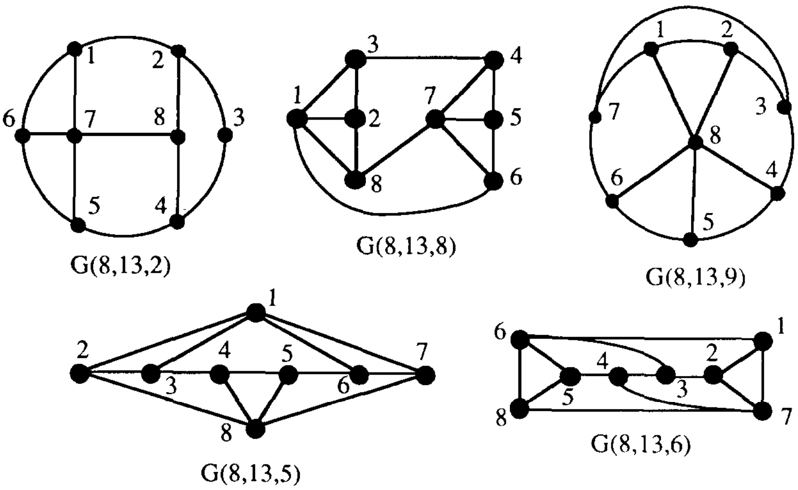

Fig. 5. Graphs $G(8,13,k)$ for $k=2$, 5, 6, 8, and 9.

result of Oporowski, Oxley, and Thomas [4] mentioned earlier, there will be some smallest integer j' for which the collection of good planar graphs that have exactly j' edges and no W_6-minor is empty. Thus we are able to conclude that all good planar graphs that have a W_6-minor have fewer than j' edges. It is routine, although tedious, to show that j' is 18. The remaining details required to establish the value of j' are similar to those used to show that j' is at least 14. Due to the amount of case-checking involved, these details have been omitted.

The diagram in Fig. 6 specifies the total number of non-isomorphic good planar graphs having h vertices, j edges, and no W_6-minor. There are a total of 231 non-isomorphic, simple, 3-connected, planar graphs that have a W_5- and no W_6-minor.

To conclude the proof of Theorem 2.2, it remains to determine which of the 231 graphs are.splitters. Recall that G is a splitter if every good planar extension of G and every good planar coextension of G has a W_6-minor. Equivalently, G is a splitter if every good planar extension of G and every good planar extension of G^* has a W_6-minor. We must now check each of the 231 graphs and determine those for which each good planar extension and good planar coextension have a W_6-minor. Although this a lengthy task, it is not difficult. Clearly every graph having 17 edges is a splitter; however, there are two other graphs that are also splitters. These graphs have 16 edges and nine vertices; each is also self-dual. Fig. 2 depicts the 38 splitters.

4. Corollaries of the main theorem

We now characterize all planar graphs G that have no W_6-minor. It is straightforward to see that G has no W_6-minor if and only if $\tilde{G}$, the simple graph associated

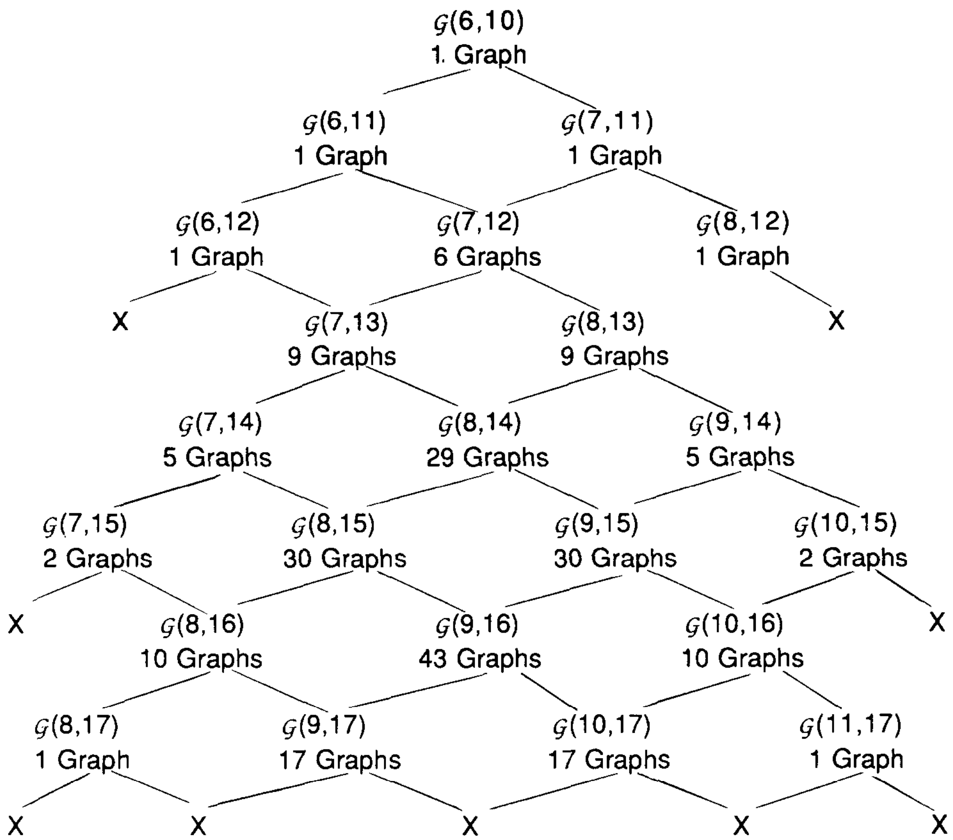

Fig. 6. Diagram of $\mathscr{G}(h,j)$.

with G, has no W_6-minor. Suppose G is not 3-connected. The following corollary states that G can be constructed from 3-connected graphs and graphs having a small number of vertices. The proof follows immediately from Corollary 1.4 and Theorem 2.2.

Corollary 4.1. *Let G be a planar graph having no W_6-minor. Suppose G has at least four vertices. Then G can be obtained from a set of graphs, $\{H_1, H_2, \ldots, H_n\}$, where, for $1 \leqslant i \leqslant n$, either H_i is a 3-connected, planar graph that does not have a W_6-minor, or H_i has three or fewer vertices, by a sequence of operations of disjoint unions, 1-sums, and 2-sums.*

Finally we give an upper bound for the number of edges of a simple, planar graph that does not have a W_6-minor.

Corollary 4.2. *Let G be a simple, planar graph having no W_6-minor. Let $v = |V(G)| \geqslant 3$ and $\varepsilon = |E(G)|$. Then*

$$\varepsilon \leqslant \lceil (14v - 27)/5) \rceil . \qquad (*)$$

Proof. We proceed by induction on v. If v is three, then G is a 3-cycle and the proof of Inequality (*) is trivial. Assume Inequality (*) is valid for all graphs with fewer than v vertices.

Suppose G is 3-connected. If G does not have a W_5-minor, then as G is isomorphic to a minor of Q_3, $K_{2,2,2}$, or H_7. Thus G has four, five, six, seven, or eight vertices. In the last three cases, G has no more than 12 edges and (*) holds. If G has five vertices, then G is a proper minor of K_5 and hence G has no more than nine edges and again (*) holds. If G has four vertices, then G is a minor of W_3 and hence G has no more than eight edges so that (*) also holds in this case. We may now assume that G has W_5-minor. Then G belongs to one of the classes of graphs $\mathscr{G}(h,j)$ and G satisfies Inequality (*).

Now suppose G is 2-connected and not 3-connected. Then G is isomorphic to $P((G_1, p_1), (G_2, p_2))\backslash p$ for some graphs G_1 and G_2 with basepoints p_1 and p_2, respectively. Since G is simple, we may assume that G_2 is simple and either G_1 or $G_1\backslash p_1$ is simple. Then $|E(G)| \leqslant |E(\tilde{G}_1)| + |E(G_2)| - 1$. For $i=1$ and 2, G_i has fewer than v vertices. Therefore, by the induction assumption, we have that

$$\begin{aligned}|E(G)| &\leqslant \lceil (14|V(G_1)|-27)/5 \rceil + \lceil (14|V(G_2)|-27)/5 \rceil - 1\\ &\leqslant \lceil (14|V(G_1)|-27+14|V(G_2)|-27)/5-1+4/5 \rceil\\ &= \lceil (14|V(G)|-27)/5 \rceil.\end{aligned}$$

The second inequality is valid since $\lceil a/5 \rceil + \lceil b/5 \rceil \leqslant \lceil (a+b+4)/5 \rceil$ for all integers a and b. Now suppose that G is not 2-connected. Let $G_1, G_2, \ldots, G_k$ be the blocks of G and suppose G has k' connected components. Then $|V(G)| = \sum_{i=1}^{k} |V(G_i)| - (k-k')$ and $|E(G)| = \sum_{i=1}^{k} |E(G_i)|$. Since $v_i = |V(G_i)|$ is strictly less than v, we may apply the induction hypothesis to G_i. Then

$$\begin{aligned}|E(G)| &= \sum_{i=1}^{k} |E(G_i)|\\ &\leqslant \sum_{i=1}^{k} \lceil (14v_i - 27)/5 \rceil\\ &\leqslant \left\lceil \sum_{i=1}^{k} (14v_i - 27)/5 \right\rceil + (k-1)\\ &= \lceil (14v - 8k - 14k')/5 - 1 \rceil\\ &\leqslant \lceil (14v-27)/5 \rceil.\end{aligned}$$

The last inequality holds since both k and k' are at least one. The proof of Inequality (*) is now complete. □

It is worth noting that there are many graphs that satisfy equality in (*). The only 3-connected graphs that satisfy equality are W_3, W_4, $K_{2,2,2}$, and any graph that is a member of $\mathscr{G}(6, 12)$, $\mathscr{G}(7, 15)$, or $\mathscr{G}(8, 17)$. Moreover, for all $n \geqslant 4$, there is a graph with n vertices that satisfies equality in $(*)$. We have seen that equality can be satisfied when $4 \leqslant n \leqslant 8$. Let $n \geqslant 9$. If $n \equiv 0$ or 4 (modulo 5), then let $m = \lfloor n/5 \rfloor$; otherwise let $m = \lfloor n/5 \rfloor - 1$. Suppose $G_1, G_2, \ldots, G_{m-1}$, and G_m are members of $\mathscr{G}(7, 15)$. The graph H will depend upon the congruence class of n modulo 5. If $n \equiv 0 \pmod 5$, then let H be $K_5 - e$. If $n \equiv 1 \pmod 5$, then let H be a member of $\mathscr{G}(6, 12)$. If $n \equiv 2 \pmod 5$, then let H be a member of $\mathscr{G}(7, 15)$. If $n \equiv 3 \pmod 5$, then let H be a member of $\mathscr{G}(8, 17)$. Finally, if $n \equiv 4 \pmod 5$, then let H be W_3. Then every parallel connection of $H, G_1, G_2, \ldots, G_{m-1}$, and G_m is a simple graph that has $|V(G)| + 5m$ vertices and $|E(H)| + 14m$ edges. It is easy to check that these parameters do satisfy equality in $(*)$.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (North-Holland, New York, 1976).

[2] G.A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs. J. London Math. Soc. 27 (1952) 85–92.

[3] S. Negami, A characterization of 3-connected graphs containing a given graph. J. Combin. Theory 32 (1982) 69–74.

[4] B.S. Oporowski, J.G. Oxley and R. Thomas, Typical subgraphs of 3- and 4-connected graphs. J. Combin. Theory Ser. B 57 (1993) 239–257.

[5] J.G. Oxley, The regular matroids with no 5-wheel minor. J. Combin. Theory Ser. B 43 (1989) 292–305.

[6] P.D. Seymour, Decomposition of regular matroids. J. Combin. Theory Ser. B 28 (1980) 305–359.

[7] W.T. Tutte, A theory of 3-connected graphs. Nederl. Akad. Wetensch. Proc. Ser. A 64 (1961) 441–455.

[8] D.J.A. Welsh, Matroid Theory (Academic Press, London, 1976).

[9] H. Whitney, Congruent graphs and the connectivity of graphs. Amer. J. Math. 54 (1932) 150–168.